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Quantitative Analysis of Team Size and its Hierarchical Structure

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Abstract

Typically, successful project teams are composed of compact groups of specialists, constituting a hierarchy. Obviously such a hierarchical construction of the project team has a deep rationale as the most desirable from the standpoint of maximizing productivity. This work is devoted to the quantitative study of the problem of justification of the hierarchical structure of project teams through the analysis of connectivity and communication between people. The study is based on the theory of Massively Interconnected Systems and the differential equation of connectivity.

Analytical dependences for describing the productivity of project teams as a function of the total number of people and their distribution in a hierarchical structure are obtained.

Key words: Interaction between the people, theory of Massively Interconnected systems, connectivity field, vector representation of connectivity fields, differential equation of connectivity, Power Laws, team size, group size, team productivity, hierarchical structure of design teams.

Introduction

Cells constitute integral part of any system, including human communities and organizations. Structures built on the basis of cells are the most effective in all respects. Human cells or human groups are powerful means for increasing efficiency of the organization of human labor.

Group structure of human work facilitates the coordination and supervision of the people daily effort, and thus improves the overall productivity.

From the practice of human work organization has long been known that for work most effective are the groups consisting of 3-7 people. The reason for this is that the increasing numbers of people in the group increases the aggregate knowledge and skills of the group which has a positive impact on the effectiveness of human work. But on the other hand an increase in the number of people in the group leads to an increase in the number of contacts between the people. This, in turn, leads to further loss of time and productivity of work. Thus the same contacts between people have both positive and negative impact on the productivity of their work.

The aggregate work quality depends on the number of people in the group and their performance indicators. This indicates that under certain conditions it is possible to find a balance between the negative and positive trends, which accompany increase in the number of people in groups. As a result, depending on the specific conditions of work, one can find the optimal number of people in groups, providing the minimal effort to carry out the work.

To accomplish such a task first of all it is necessary to describe quantitatively the interaction between human beings and their connectedness in time and space. It is very important to know how often and how long the people are interacting. What are the physical distances between them during the work? If they are in close quarters, what is their relative position in the workplace?

Qualitative analysis of these issues is reflected in the literature in great details [1]. But the quantitative aspects of people interaction have been insufficiently explored in the literature and this work is dedicated to filling some aspects of this gap.

Contacts between the people and their connectivity analysis

Typically, the quantitative description of human contacts is reduced to the use of a simple combinatorial formula [2, 3]

$$C = \frac{N(N-1)}{2} \qquad (1)$$

Here C is the number of contacts between N people.

This relationship assumes the same connectedness between the people, regardless of their number N. It is very important to notice that this is true only for the small human groups. In reality, the connectivity between people is uneven, which strongly affects the overall number of contacts C. Therefore more realistic assessment of C needs new approaches to the mathematical modeling of the connectedness of the people in design teams, because the formula (1) overestimates the number of contacts.

This work presents a new approach to the problems of people interaction quantitative analysis.

This new approach to the quantitative description of the connectedness of people is based on the theory of Massively Interconnected Systems [4, 5].

The essence of this approach to the problems of quantitative analysis of the project team is that the totality of the people in it is considered as a Massively Interconnected System.

Massively Interconnected Systems

Massively Interconnected systems (MIS) can be characterized by a large number of elements and massive interconnections between them [4, 5]. Such systems are to be found everywhere. Among the examples of MIS are the human society, Internet and World Wide Web, an organization, a human group, a design team, a communication system or some semiconductor chip with a millions of elements and interconnections between them. The quantitative description of such systems becomes imperative nowadays because of the variety of potential applications of the latter. In organization science those practical applications are related to the structural analysis and synthesis of the organization and development of mathematical theory of the projects, including quantitative analysis of the behavior of human groups.

From the mathematical point of view such systems can be divided into static and dynamic Massively Interconnected Systems. While in static massively interconnected systems the connections between elements are time independent, in dynamic massively interconnected systems they are time dependent.

Graphical Representations of Massively Interconnected Systems

Let's start with simple graphical representations of massively interconnected systems, having a goal to quantitatively describe the spatially distributed elements of the latter and the massive connections among them. For that purpose it is useful to examine the common behavior for separate elements and their connections in such systems. Each element in the system has chaotic connections with the other elements. Elements of a massively interconnected system and connections of one of those elements are shown in the Fig.1.

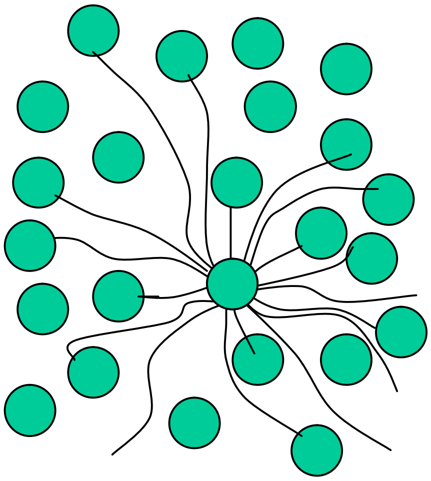

Fig.1. One element of a MIS along with its connections with other elements

Separating the element out along with its connections we will have the picture presented in Fig.2.

The same may be done with the other elements and as a result we will always obtain almost the same picture where an element emits a certain number of connections which are gradually absorbed by the other elements. This chaotic penetration of interconnects into the environment with some variations is common for all massively interconnected systems.

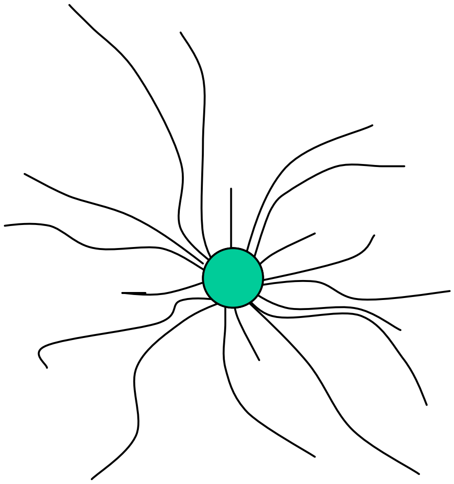

Fig.2: A separate element along with its connections

Therefore it may become a focus around which we can concentrate our modeling effort. Another characteristic for such a behavior is that the connection penetration image into the environment is very close to the well-known diffusion process. Thus for the same modeling purposes it is useful to make parallels between the diffusion and the process of connection penetration into the environment.

Spatial Densities of Interconnections and Elements

The study of an ordinary interconnection image in MIS shows that quantitatively it can be characterized by the spatial densities of elements and connections. If the MIS has a hierarchical structure with different levels of subsystems or elements, it may be described in any level of physical hierarchy. Irrespective to the level of physical hierarchy, the MIS will represent itself as a spatial conglomeration of elements and connections. The spatial densities of elements can be considered as scalar quantities. But the connections cannot be characterized only by the spatial densities, because in each point of the connectivity field they have another important characteristic – direction, which must be accounted in connectivity mathematical models.

Vector Representation of Connectivity fields as a necessity

Let’s consider a simple connectivity field in the form of parallel lines (Fig.3.). In order to reveal the vector behavior of such a connectivity field let’s conduct a simple geometrical experiment. Let’s consider three line segments in the field which have different angles with the direction of the parallel lines. The goal of the experiment is to find the number of crossings of the line segments with the connectivity field that has a constant density $\vec{\mu}$. Quantitatively the vector $\vec{\mu}$ is equal to the number of crossings with the field of the unit line segment, which is perpendicular to the connections. Obviously the number of crossings with the line segment will have its maximum value when it is perpendicular to the connections (segment 1 in the Fig.3.).

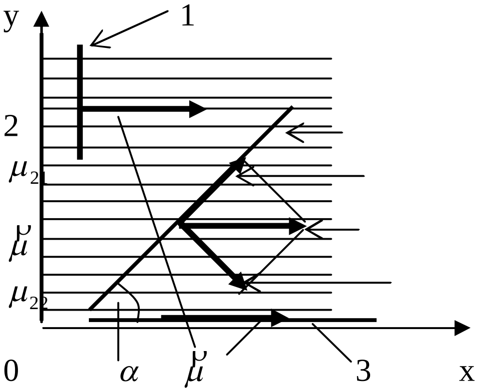

Fig. 3: Homogeneous Vector Field of connections

The number of crossings will have a value 0 when line segment is parallel to the connections (segment 3). In order to solve the problem in the general case (segment 2) when the line segment has an arbitrary angle with the connections (or X axis) it is necessary to consider the connection field as a vector field. For that it is necessary to split the vector $\vec{\mu}$ into two components, μ_{21} and μ_{22}. The first one is parallel to the arbitrary line segment and the component μ_{22} is perpendicular to the same segment. Therefore the number of crossings of the line segment L with the connection field is equal to

$$m_2 = \mu_{22} L = L \|\vec{\mu}\| \sin\alpha \qquad (2)$$

These calculations show that in order to account for the directions of connections in the connectivity field the latter has to be considered as a vector field.

Generic Connectivity Fields

Consider the generic connectivity field shown in Fig.4. Each point of this field can be characterized by a vector $\vec{\mu}(x,y)$ that has the direction of the tangent to the connections. The magnitude of this vector $\|\vec{\mu}\|$ is the density of the connections in the point (x,y).

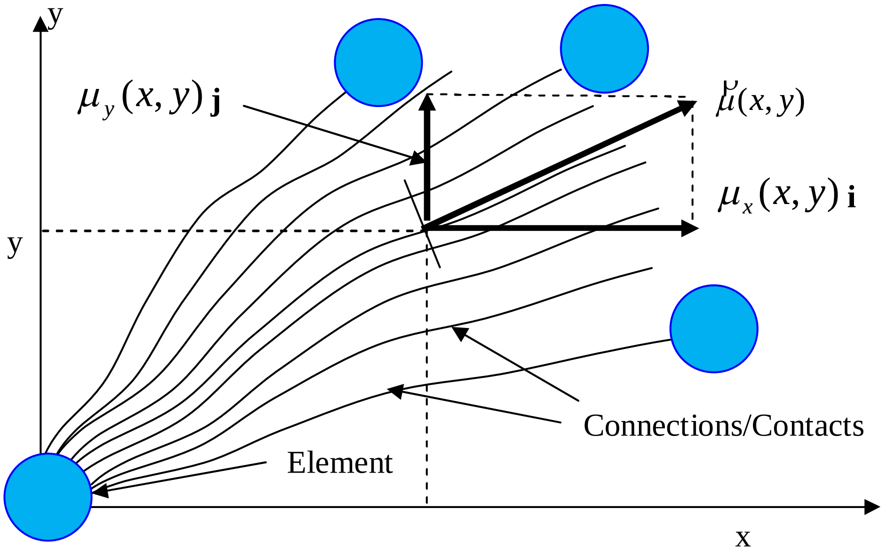

Fig.4 Definition of the generic connectivity field

For each point of the field the connectivity density vector can be split up into the components μ_x and μ_y.

$$\vec{\mu}(x,y) = \mu_x(x,y)\,\mathbf{i},+\mu_y(x,y)\,\mathbf{j}, \qquad (3)$$

where **i** and **j** are the standard basic vectors. The norm of the density vector is

$$\|\vec{\mu}\| = \sqrt{\mu_x{}^2 + \mu_y{}^2} \qquad (4)$$

Differential Equation of Connectivity (Steady State case)

Based on the fact that the number of elements and connections between them for a certain part of the connectivity field are strongly interrelated, let's explore the balance between them in the vicinity of an arbitrary point (x, y) (Fig.5). For doing that let's consider a small area $ds = dxdy$. Connections are incoming into that area from the left and bottom sides of the area

ds and are outgoing from the top and right sides of the same area. The number of connections m_{in} that enters into the small area ds from the left side can be determined as

$$m_{inl} = \mu_x(x, y)dy. \tag{5}$$

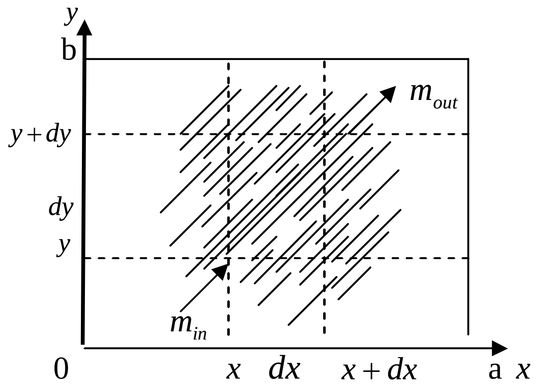

Figure 5: Balance between the connections and contacts of elements in the small area $ds = dxdy$

Similarly, the number of outgoing connections m_{outr} from the right side of the same small area ds can be calculated by the formula

$$m_{outr} = \mu_x(x + dx, y)dy. \tag{6}$$

Thus the difference between the numbers of incoming and outgoing connections in the direction of "x" axis equals to

$$m_{outr} - m_{inl} = [\mu_x(x + dx, y) - \mu_x(x, y)]dy. \tag{7}$$

Similarly we can find the number of incoming connections from the bottom of the small area ds as

$$m_{inb} = \mu_y(x, y)dx. \tag{8}$$

The number of outgoing connections m_{outt} from the top of the same small area ds can be calculated by the formula

$$m_{outt} = \mu_y(x, y + dy)dx. \tag{9}$$

Thus the difference between the numbers of incoming and outgoing connections in the direction of “y” axis equals to

$$m_{outt} - m_{inb} = [\mu_y(x, y + dy) - \mu_y(x, y)]dx . \qquad (10)$$

So the total difference between the numbers of incoming and outgoing connections for the small area ds can be calculated as the sum of the differences (7) and (10)

$$m_{out} - m_{in} = m_{outr} - m_{inl} + m_{outt} - m_{inb} . \qquad (11)$$

Substituting the expressions (7) and (10) into (11) we will have

$$\begin{aligned} m_{out} - m_{in} &= [\mu_y(x, y + dy) - \mu_y(x, y)]dx + \\ &+ [\mu_x(x + dx, y) - \mu_x(x, y)]dy . \end{aligned} \qquad (12)$$

Let’s analyze this expression assuming that the density of connections is a continuous function in the small area ds and in its vicinity. Based on that we can represent $\mu_x(x + dx, y)$ as a power series by dx

$$\mu_x(x + dx, y) = \mu_x + \frac{\partial \mu_x}{\partial x} dx + \ldots \qquad (13)$$

Substituting (13) into the (12) and neglecting the higher order terms we have

$$m_{out} - m_{in} = \frac{\partial \mu_x}{\partial x} dxdy . \qquad (14)$$

Similarly representing $\mu_x(x, y + dy)$ as a power series by dy we have

$$m_{outt} - m_{inb} = \frac{\partial \mu_y}{\partial y} dxdy . \qquad (15)$$

Substituting the results obtained from (13) and (15) into the expressions (11) or (12) we will have

$$m_{out} - m_{in} = (\frac{\partial \mu_x}{\partial x} + \frac{\partial \mu_y}{\partial y})dxdy . \qquad (16)$$

As it can be seen from Fig.5, the difference between the numbers of incoming and outgoing connections is formed as a result of the termination of a portion of incoming connections in the small area and generation of a portion of outgoing connections in the same small area. In order to account for that termination and generation process of connections let's introduce the notions of densities of connection sinks $\rho_s(x,y)$ and connection sources $\rho_n(x,y)$ in the connectivity field. Based on these notions we are able to calculate the same difference $m_{out} - m_{in}$ by the following expression:

$$m_{out} - m_{in} = [\rho_n(x,y) - \rho_s(x,y)]dxdy \ . \qquad (17)$$

Equating the expressions (16) and (17) we will have

$$\frac{\partial \mu_x}{\partial x} + \frac{\partial \mu_y}{\partial y} = \rho_n(x,y) - \rho_s(x,y) \ , \qquad (18)$$

which represents the above-mentioned balance between the elements and connections in the form of a differential equation. The functions $\mu_x(x,y)$ in the point $x = 0$ and $\mu_y(x,y)$ in the point $y = 0$ can serve as boundary conditions for the equation (18), that is to say - functions $\mu_x(0,y)$ and $\mu_y(x,0)$ in the left and bottom edges of the wiring field.

Other Forms of the Connectivity Differential Equation

Because the elements in the connectivity field are the main sinks and sources of the connections, it is evident that the equation (18) contains information about the elements as well, but in an indirect (implicit) form. In order to deal with the spatial densities of elements $n(x,y)$ let's denote the numbers of the connection sources and sinks for the elements in the vicinity of the point (x,y) by $p(x,y)$ and $q(x,y)$. It is obvious that

$$\rho_n(x,y) = p(x,y)n(x,y) \qquad (19)$$

and

$$\rho_s(x,y) = q(x,y)n(x,y) \ . \qquad (20)$$

Substituting (19) and (20) into the main equation (18) we can have another form for the latter.

$$\frac{\partial \mu_x}{\partial x} + \frac{\partial \mu_y}{\partial y} = [p(x,y) - q(x,y)]n(x,y) \ . \qquad (21)$$

From mathematical point of view it is convenient to incorporate the components of the density vector of connectivity directly into the right hand side of the equation (18). In order to do that let's introduce two new notions: connection termination intensity $\gamma(x,y)$ and connection generation intensity $\beta(x,y)$ such that $\gamma(x,y)dxdy$ is the probability of the termination of one connection in the area ds, and $\beta(x,y)dxdy$ is the probability of origination of one connection in the area ds. Based on that we can define the connection sink and source densities by the following formulas:

$$\rho_s(x,y)=\gamma(x,y)(\mu_x+\mu_y) \tag{22}$$

and

$$\rho_n(x,y)=\beta(x,y)(\mu_x+\mu_y). \tag{23}$$

Substituting (22) and (23) into the main equation (18) we can have a very useful form of the differential equation of connectivity.

$$\frac{\partial\mu_x}{\partial x}+\frac{\partial\mu_y}{\partial y}=[\beta(x,y)-\gamma(x,y)](\mu_x+\mu_y) \tag{24}$$

This equation with corresponding boundary conditions is able to describe arbitrary massively interconnected systems.

Let's apply the equation (24) for connectivity analysis of one-dimensional cases of spatially distributed elements. The main goal here is to describe the spatial connectivity of a stand-alone element. Then the next step is to solve the differential equation for that element and find its connectivity field. If a system incorporates some number of elements, then applying the solution for one element and the principle of superposition of the connectivity field we can find connectivity characteristics for the arbitrary configuration or shape of connectivity fields. This approach is valid for both homogeneous and heterogeneous connectivity systems.

Differential Equation for Separate Elements: One-dimensional case

Let's consider a simple homogeneous connectivity system in the form of an infinite chain of elements positioned along the "x" axis in the points with integer values. Each element has α terminals half of which are connected with the left-hand side elements and another half of the terminals are connected with the elements on the right-hand side (Fig.6.).

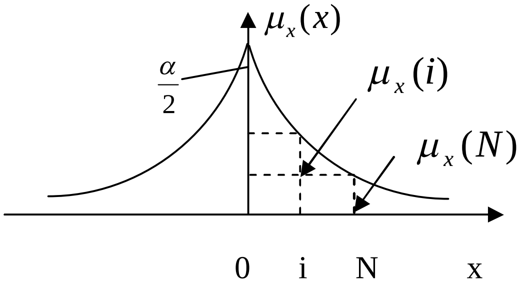

Fig.6. Left and right branches of the connectivity function $\mu_x(x)$

All the elements in such a system have the same connectivity characteristics (Fig.7) and the system can be described by connectivity equation (24), with $\mu_y = 0$

$$\frac{\partial \mu_x}{\partial x} = [\beta(x) - \gamma(x)]\mu_x \, , \qquad (25)$$

where all parameters are the functions of “x” variable only. In this equation μ_x represents the number of connections, length of which exceeds “x”.

Let’s separate the element positioned in the point $x = 0$ along with its connections. In this case connection origination function $\beta(x)$ transforms into a boundary condition $\mu_x(0) = \frac{\alpha}{2}$, and the equation itself has the form

$$\frac{\partial \mu_x}{\partial x} = -\gamma(x)\mu_x \, . \qquad (26)$$

Since by definition $\mu_x(x)$ is the number of connections in the section “x” that have been originated from element “0”, the ratio $\mu_x(x)/\frac{\alpha}{2}$ will be the probability that the length of connections of that element will exceed “x”.

Consequently we can define the connection length distribution function as

$$F(x) = 1 - \frac{2\mu_x}{\alpha} \qquad (27)$$

The solution of the equation (26) has the form

$$\mu_x(x) = \frac{\alpha}{2} Exp[-\int_0^x \gamma(y)]dy \,, \qquad (28)$$

which describes the right branch in Fig.6 and is directly related with the connection length distribution function F(x), which can be described as

$$F(x) = 1 - Exp[-\int_0^x \gamma(y)]dy \qquad (29)$$

Combining the solution (28) with the principle of superposition of the connectivity field, we can calculate connectivity characteristics for the group of “N” elements (Fig.7). Assume we need to estimate the number of terminals for that group of elements (each one of those has the same connectivity function) with the other neighboring elements.

Due to the symmetric character of the connectivity functions, we can calculate the number of terminals outgoing from the right side of the group only - $m_r(N)$. Besides, we can represent the number of terminals as the sum (superposition) of the number of terminals for each separate element. The number of terminals for an arbitrary element “i” will be the value of the function $\mu_x(x-i)$ in the point N, i.e. $m_i = \mu_x(N-i)$.

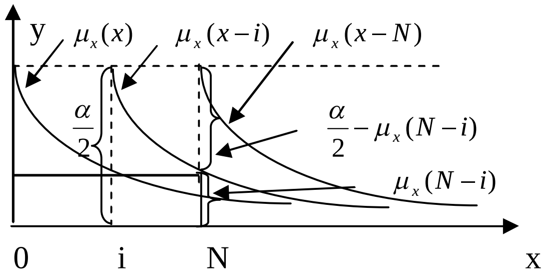

Fig.7. Group connectivity exploration

Thus the total number of terminals $m(N)$ can be calculated as the following sum

$$m(N) = 2m_r(N) = 2\sum_1^N m_i = 2\sum_1^N \mu_x(N-i) \,. \qquad (30)$$

This is the discrete analogue of the well known Rent’s Rule [7] for the number of connection terminals or contacts as a function of N.

For the large number of elements we can make a transition to the continuous analogue of the same rule

$$m(N) = 2\int_0^N \mu_x (N - x)dx\,. \qquad (31)$$

Differentiating relationship (31) with regard to the total number of elements N we can have

$$\frac{dm(N)}{dN} = 2\mu_x(N)\ . \qquad (32)$$

This expression establishes a functional relationship between connectivity characteristics of the system and connectivity characteristics of the element μ_x. Taking into account the relationship (27) between $\mu_x(x)$ and connection length distribution function $F(x)$ we will have

$$\frac{dm(N)}{dN} = \alpha[1 - F(N)]\,, \qquad (33)$$

which establishes a functional relationship between the number of terminals of a homogeneous one-dimensional system and connection length distribution function.

Power laws as specific solutions of the connectivity differential equation

To elucidate the relationship between differential equation of connectivity and power laws let's analyze the solution of one-dimensional equations in general form (28) and distribution function (29).

Such a choice of one-dimensional equation makes it possible to simplify the analysis without losing common consideration.

As can be seen from these expressions, the behavior of connectivity depends on the sole function of the extinction of the connections $\gamma(x)$. Therefore, in practice the variety of connectivity between subsystems can be identified by this function.

On the other hand, these functions are easy to be studied for different systems in an experimental way with the resulting approximating formulas as specific cases of $\gamma(x)$. These specific approximating forms of $\gamma(x)$ allow to continue theoretical analysis of connectivity for various systems. Let's analyze some specific forms of the connectivity extinction function $\gamma(x)$.

1. $\gamma(x) = \gamma$ = Const.

For this particular case from (28) and (29) we can have

$$\mu_x(x) = \frac{\alpha}{2} Exp(-\gamma x) \qquad (34)$$

$$F(x) = 1 - Exp(-\gamma x) \qquad (35)$$

This means that for $\gamma(x) = \gamma$ = Const connection length has an exponential density function

$$f(x) = \frac{dF(x)}{dx} = \gamma Exp(-\gamma x) \qquad (36)$$

2. Hyperbolic extinction function

$$\gamma(x) = \frac{A}{x}, \qquad (37)$$

where A is an approximation constant. Substituting (37) into the expressions (28) and (29) we can have

$$\mu_x(x) = \frac{\alpha}{2} Exp[-A\int_0^x \frac{1}{y} dy] \qquad (38)$$

$$F(x) = 1 - Exp[-A\int_0^x \frac{1}{y} dy] \qquad (39)$$

After integration we can have

$$\mu_x(x) = \frac{\alpha}{2}(1-A)x^{-A} \qquad (40)$$

$$F(x) = 1 - (1-A)x^{-A} \qquad (41)$$

From here for the connection length density function we can have

$$f(x) = \frac{dF(x)}{dx} = A(1-A)x^{-A-1} \qquad (42)$$

The formulas (40), (41) and (42) reflect the fact that in most cases every person is communicating with the immediate environment and the connectivity is quickly weakened with the increasing distance between the people. Function $\mu_x(x)$ reflects the absolute number of contacts depending on the degree of remoteness of the people.

This function is a typical example of a power law which is widely applied in various fields of human knowledge. Distribution function (41) and density function (42) reflect the same behavior of decreasing number of contacts between the people as a function of their distance. Thus, for a certain class of extinction functions $\mu_x(x)$ the power laws are the specific solutions of the differential equation of connectivity.

The number of contacts within human group and the number of contacts/terminals of a human group with the environment

If each element in average has α contacts with all other elements then the aggregate number of contacts of N elements will be αN. If the number of terminals of N element is $m(N)$ then the number of the contacts within human group will be

$$C(N) = \alpha N - m(N) \qquad (43)$$

Taking into account the expression (32) we can find that

$$m(N) = 2\int_0^N \mu_x(y)dy \qquad (44)$$

Substituting the expressions for $\mu_x(x)$ from (34) and (40) into the expression (44) we can have the estimates for the number of terminals for two specific cases.

1. $m(N) = 2\int_0^N \mu_x(y)dy = \alpha\int_0^N Exp(-\gamma y)dy = \frac{\alpha}{\gamma}(1 - Exp[-\gamma N]) \qquad (45)$

2. $m(N) = 2\int_0^N \mu_x(y)dy = \alpha(1-A)\int_0^N y^{-A}dy = \alpha N^{1-A} \qquad (46)$

Substituting these results into the expression (41) we can obtain formulas for the number of contacts within human group consisting of N person

1. $C(N) = \alpha N - \frac{\alpha}{\gamma}(1 - Exp[-\gamma N])$ (47)

2. $C(N) = \alpha N - \alpha N^{1-A}$ (48)

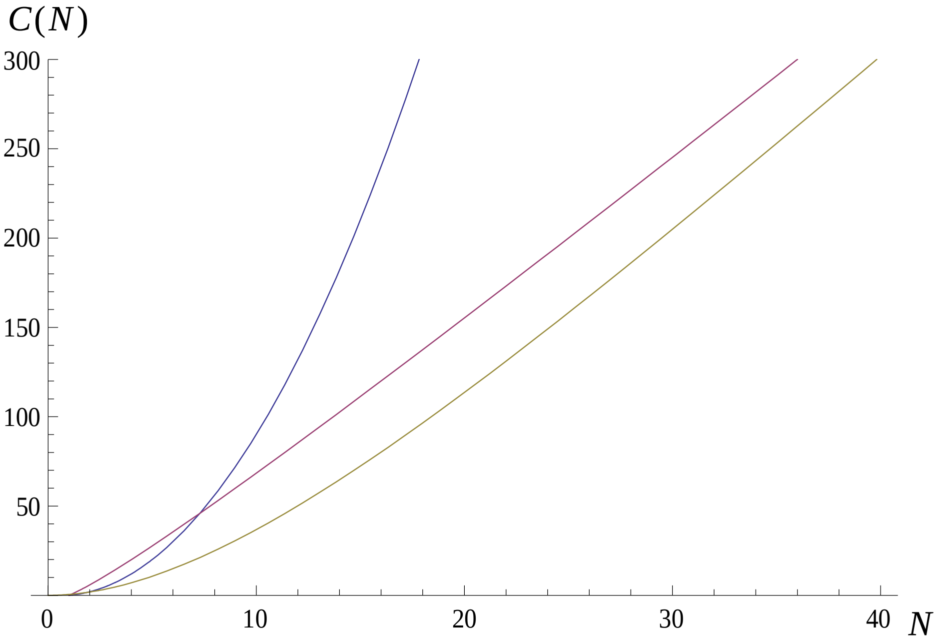

Fig.8 Comparison of three formulas for the estimation of the number of contacts as a function of the number of people

The analysis of formulas (47) and (48) for estimating the number of contacts within the group of people shows that the dependence of the number of contacts from the number of people in the group is essentially linear in nature, which is very different from formula (1), where this dependence is quadratic in nature [Fig.8].

Rent's Rule as the specific solution of the connectivity differential equation

In 1960 E.F. Rent of IBM published an internal memorandum that contained a relationship between the number of external signal connections or terminals of a logic block n_c and number of logic elements N.

$$n_c = \alpha N^r \quad (49)$$

Here, α is proportionality constant and r is the Rent's exponent. The values of α and r for the IBM computers were reported to be 2.5 and 0.6, respectively [7].

The same rule can be applied for estimating the number of internal connections between the elements [4]

$$n_{in} = \alpha(N - N^r) \qquad (50)$$

The analysis shows that there is a close link between the Rent's rule and differential equation of connectivity. Comparing the formulas for the number of internal contacts (46) and terminals (48) derived from the solutions of differential equations of connectivity for $\gamma(x) = \frac{A}{x}$ with the Rent's rule, it is easy to see that when $r = 1 - A$ there is complete agreement between them.

This comparison shows that the Rent's rule in addition to electronics can also be applied in other areas, including social networks, Internet and other Massively Interconnected systems. According to this rule the number of internal connections and terminals of human groups can be described by the formulas (49) and (50). For this kind of applications the coefficient α presents the total number of internal and external contacts for one person. Exponent r indicates the ratio between internal and external contacts. The large exponent r means a small number of internal contacts and a large number of external contacts. There are two limit cases for $r = 0$ and $r = 1$.

For the case of the $r = 0$, all contacts are internal, and for the $r = 1$, all contacts are external. Below the formulas (49) and (50) will be used to analyze the behavior of the project team and its hierarchy.

Team productivity vs. team size and group size

Consider a working team consisting of N individuals (Fig.9).

The team divided into groups consisting of n_0 persons, including the group leader. This means that the team has $\frac{N}{n_0}$ groups.

Let's apply the obtained power laws of interpersonal contacts, to analyze the productivity of the team with a hierarchical structure.

During this analysis are used the following denotations:
W – is the amount of work performed during the time t,
P_{01} and P_{02} are the productivities of the workers of the first and second levels of the hierarchy.

W_1 and W_2 are the amounts of work performed by the workers of the first and second levels of the hierarchy during the same period of time.

It is clear that

$$W = W_1 + W_2 \tag{51}$$

Let's define W, W_1 and W_2 for the team hierarchy separately.

$$W = t * N * P(N, n_0)\,, \tag{52}$$

Here $P(N, n_0)$ - is the average aggregate productivity of one person for the whole team.

$$W_1 = P_{01}(N - \frac{N}{n_0})(t - t_1)\,, \tag{53}$$

where $(N - \frac{N}{n_0})$ - is the number of the first level workers without leaders,

t_1 - This is the time loss of one first level employee because of his contacts with others.

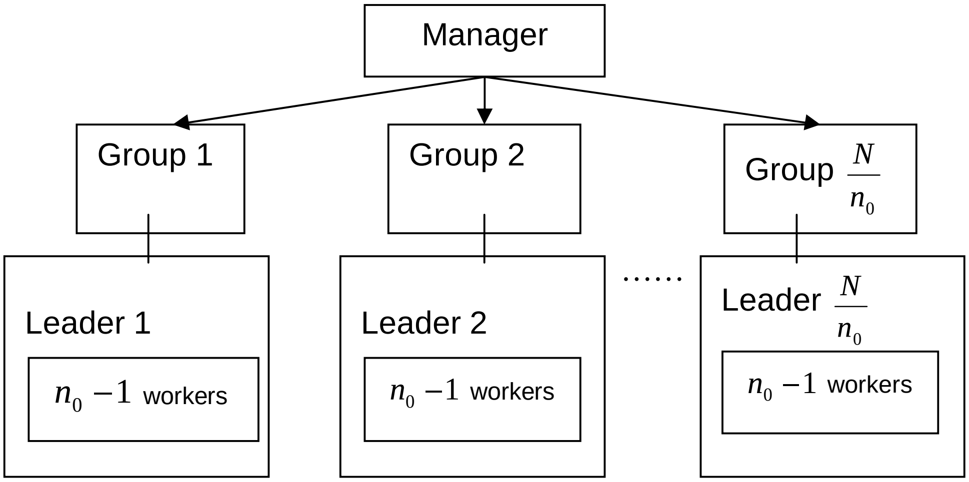

Fig. 9. Hierarchical structure of the design team

In its turn t_1 can be found using the Rent's rule

$$t_1 = \frac{\alpha_1 t_{k1}(n_0 - n_0^{r_1})}{n_0}\,, \tag{54}$$

where t_{k1} - is the average duration of one contact at the first level of communication;
r_1 - is the Rent's exponent for the first level of communication.
Thus for W_1 we finally can have the following expression

$$W_1 = P_{01}(N - \frac{N}{n_0})[1 - \frac{\alpha_1 t_{k1}(n_0 - n_0^{r_1})}{n_0}] \tag{55}$$

Let's now find the value of W_2.

$$W_2 = \frac{P_{02}N}{n_0}(t - t_1 - t_2) \tag{56}$$

where t_1 is the loss of one leader of the group at the first level of communication and t_2 is his loss of time on the second level of communication. As we have already identified t_1, now find t_2.

$$t_2 = \frac{\alpha_2 t_{k2}[\frac{N}{n_0} - (\frac{N}{n_0})^{r_2}]}{\frac{N}{n_0}} \tag{57}$$

where t_{k2} - is the average duration of one contact at the second level of communication;
r_2 - is the Rent's exponent for the second level of communication.

From here substituting the expressions (54) and (57) into the (56), we can have

$$W_2 = \frac{P_{02}N}{n_0}\{t - \frac{\alpha_1 t_{k1}(n_0 - n_0^{r_1})}{n_0} - \frac{\alpha_2 t_{k2}[\frac{N}{n_0} - (\frac{N}{n_0})^{r_2}]}{\frac{N}{n_0}}\} \tag{58}$$

Now we can substitute W, W_1 and W_2 from (52), (55) and (57) respectively into the main balance equation (51), As a result we can have

$$N*t*p(N,n_0)=P_{01}(N-\frac{N}{n_0})[t-\frac{\alpha_1 t_{k1}(n_0-n_0^{r_1})}{n_0}]+ \qquad (59)$$
$$+\frac{NP_{02}}{n_0}[t-\frac{\alpha_1 t_{k1}(n_0-n_0^{r_1})}{n_0}-\frac{\alpha_2 t_{k2}(\frac{N}{n_0}-(\frac{N}{n_0})^{r_2})}{\frac{N}{n_0}}]$$

Dividing both parts of (59) by $N*t$, we get a new expression for the team productivity as a function of team size and group size.

$$p(N,n_0)=P_{01}(1-\frac{1}{n_0})[1-\frac{\alpha_1 t_{k1}(n_0-n_0^{r_1})}{tn_0}]+ \qquad (60)$$
$$+\frac{P_{02}}{n_0}[1-\frac{\alpha_1 t_{k1}(n_0-n_0^{r_1})}{tn_0}-\frac{\alpha_2 t_{k2}(\frac{N}{n_0}-(\frac{N}{n_0})^{r_2})}{t\frac{N}{n_0}}]$$

As we can see, team productivity is a function of some other parameters as well. In particular, the ratios $\frac{t_{k1}}{t}$ and $\frac{t_{k2}}{t}$ show the parts of work time spent on contacts of the first and second levels of the hierarchy. Two parameters α_1 and α_2 indicate the intensity of contacts between the people. In Fig.10 is presented team productivity as a function of the group size, keeping the team size and other parameters constant.

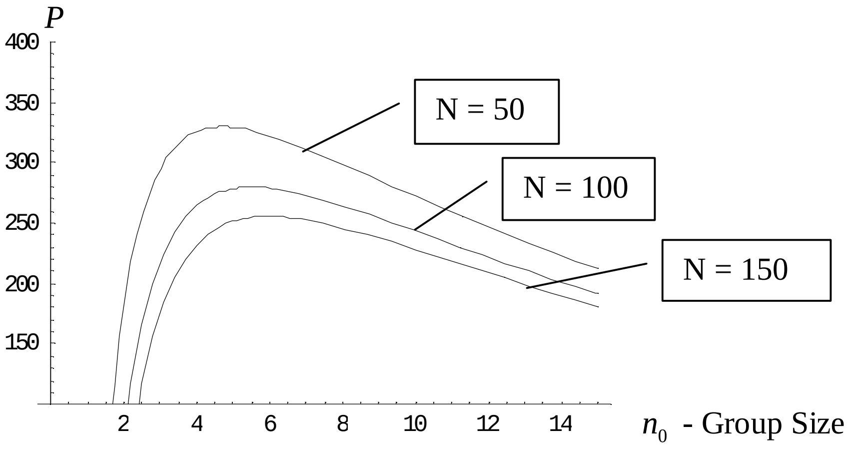

Fig.10. Team productivity vs. Group Size for constant Team Sizes

Analysis of this graph indicates that there is an optimum size of the group for which the productivity of the team has a maximum. This result is remarkable because it has been inferred solely on the considerations about the contacts between the people. This means that based on such considerations, one can synthesize the hierarchical structure of large teams.

Besides this graph shows that for the small team sizes team productivity is higher than for large teams. Also larger values of team size corresponds a larger value of the optimal group size.

In Fig.11 it is shown the relationship $P(N, n_0 = Const)$ for $n_0 = 5$, $n_0 = 10$ and $n_0 = 20$.

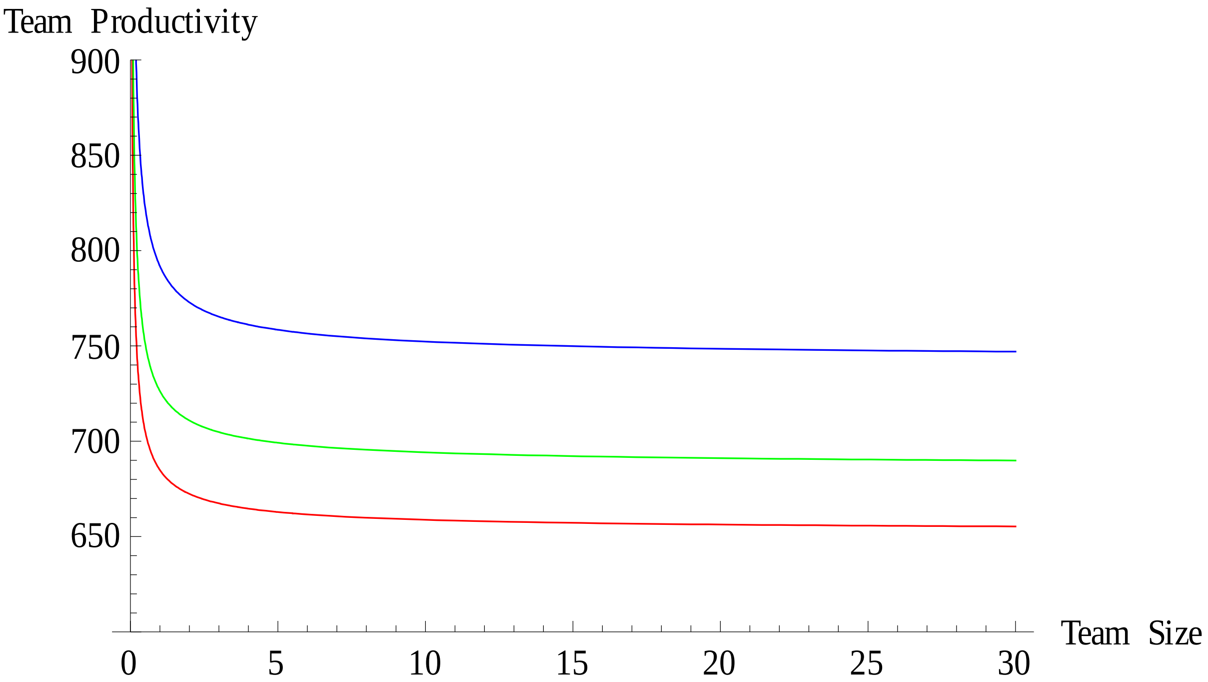

Fig.11 Team productivity vs. team size for constant values of group size

Also it is very interesting to compare this family of curves with the same family of curves from [8].

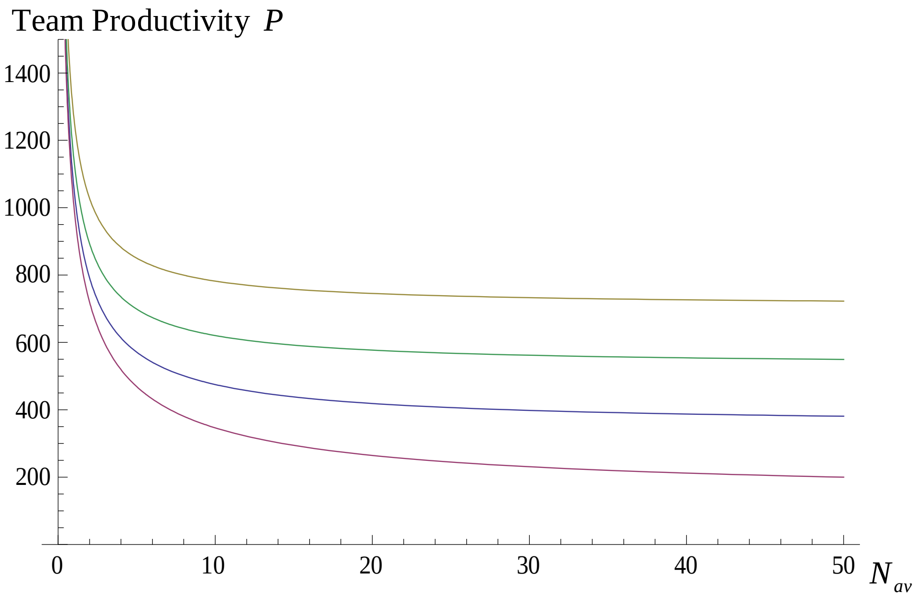

Fig.12 Team productivity vs. project average staffing level for different constant values of project parameters (Fig.4 from [8])

They are almost identical, although the derivation of these results was based on entirely different considerations. Derivation of the first of them is based on the thermodynamic type of top-down considerations [8], while the derivation in this work is based on the statistical mechanics type bottom-up considerations.

Conclusions

1. Interaction between people is a considerable part of the working time and therefore can greatly influence the productivity of work.
2. For this reason, the mathematical model for evaluating the quantitative characteristics of human contacts is important for the analysis of team productivity.
3. The simple combinatorial model, which currently is in use for this purpose is not in a position to accurately reflect the essence of the problem.
4. The solution to this problem can be found in the use of the theory of massively interconnected systems and differential equation of connectivity for the mathematical modeling of human contacts.
5. In reality contacts between the people have uneven character and can be modeled by power laws.
6. In their turn power laws are the specific solutions of the connectivity differential equation.
7. In particular, the well known in computer design Rent’s rule is a consequence of a power law solution of the differential equation of connectivity.

8. Rent's rule can be used for modeling of social networks and in particular it is a solid basis for team productivity analysis.
9. Team Performance is very sensitive to the Group Size.
10. Team Performance has maximum as a function of Group size.
11. Team Size almost has no influence on the optimal group size.
12. New models of productivity built on the basis of the theory of massively interconnected systems can be used for early project simulations and project planning.
13. Using people interaction related considerations only it is possible to synthesize the hierarchical structure of the large design teams.

Future work is connected with:

1. Analysis of power laws as specific solutions of the connectivity differential equations.
2. Analysis the hierarchical structure of organizations based on the communication type of considerations.
3. Accounting for the learning in the team productivity models.
4. Applying MIS theory and connectivity differential equation for mathematical modeling of Internet and World Wide Web.

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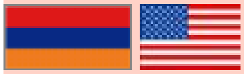

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